Boundedness of Pluricanonical maps (following Hacon – McKernan)

Theorem: Let n be some positive integer. There exists r_n such that for any smooth proj. variety X of general type the map induced by $|rK_X|$ is birational (Chr X big) for any $r \geq r_n$.

Recall: $r_1 = 3$ (easy)

$r_2 = 5$ (Bombieri)

Theorem: X smooth, general type, dim n, and say we know the above theorem in dim $< n$, and set

$$s = \left(\left\lfloor \frac{n}{\mathrm{vol}(X)^{1/n}} \right\rfloor + 2\right)(n r_{n-1} + 1).$$

Then $|rK_X|$ is birational for $r \geq 4s(r_{n-1}+1) + r_{n-1}$

Lemma: Let X be a smooth proj. variety. Say for all $m \geq m_0$ $h^0(m\omega_X) \geq 2$. Say F is a general fiber of the pencil $X \overset{|m_0\omega_X|}{\dashrightarrow} \mathbb{P}^1$. If $|s\omega_F|$ is birational, then $|t\omega_X|$ is birational for $t \geq m_0(2s+2)+s$

Example: Say ω_X is ample (/big & nef), and say $D \in |r\omega_X|$. D is of general type. Say $h^0(s\omega_D) \geq 2$.

$$0 \to \mathcal{O}_X((sr+s-r)\omega_X) \to \mathcal{O}_X((sr+s)\omega_X) \to \mathcal{O}_D(s\omega_D) \to 0$$

$s\omega_D = s(\omega_X + r\omega_X)$

$H^1(\underbrace{(sr+s-r)}_{\geq 2}\omega_X) = 0$ by $\omega(V)$-vanishing

So, can lift global sections of $H^0(s\omega_D)$ to $H^0((sr+s)\omega_X)$.

two problems: • how to get bound on r

• what if K_X is just big?

solution
$\longrightarrow$ use log canonical centers.

Definition: Let (X, Δ) be a log pair. A closed subvariety $V \subset X$ is a log canonical center if there's a log resolution $\pi : Y \to X$, w/ $\pi(E) = V$ for E an irred. divisor, and if we write $K_Y + \Gamma = \pi^*(K_X + \Delta)$, then $\mathrm{coeff}_E(\Gamma) \geq 1$.

If (X, Δ) is log canonical at generic pt of V, we say V is a pure lc center; if E is the only divisor w/ coeff $\Gamma \geq 1$ dominating V, V is an exceptional lc center.

Facts: Let (X,Δ) be a pair & $x \in X$ a lct point of X.

1) if $W_1, W_2 \in LCS(X,\Delta,x)$, then any irred. comp W of $W_1 \cap W_2$ through x is in $LCS(X,\Delta,x)$. Thus, $LCS(X,\Delta,x)$ contains a unique minimal component V.

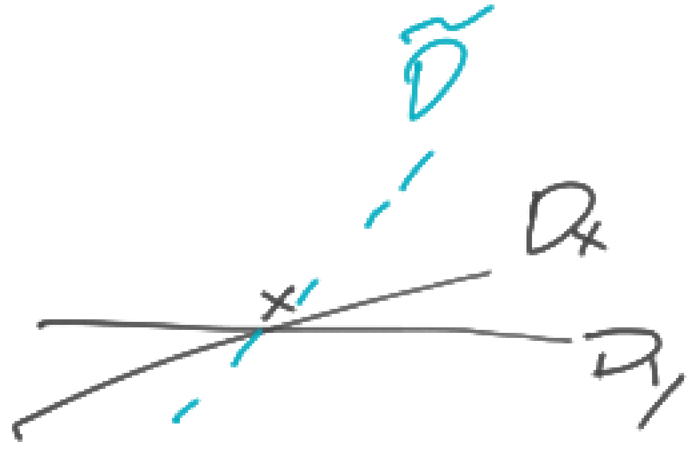
2) $\exists$ an eff. $\mathbb{Q}$-divisor E s.t.

$$LCS(X,(1-\varepsilon)\Delta+\varepsilon E, x) = \{V\} \quad \text{for } 0<\varepsilon<<1$$

3) we can assume there's a unique lc place E lying over V.

Ex: $X = \mathbb{A}^2$, $\Delta = D_x + D_y$ the union of x & y-axis

$$LCS(X,\Delta,x) = \{D_x, D_y, x\}$$



$$LCS(X,(1-\varepsilon)\Delta+\varepsilon(2\tilde{D}),x) = \{x\}.$$

Lemma: Let $\dim X = n$, $x \in X$ a smooth point, and say D is a big $\mathbb{Q}$-divisor. For every k large & divisible, $\exists\ A \in |kD|$ such that $\operatorname{mult}_x(A) > k(\operatorname{Vol}(D))^{\frac{1}{n}}$.

In particular, if $\lambda \in \mathbb{Q}$, $\lambda > \dfrac{n}{\operatorname{Vol}(D)^{\frac{1}{n}}}$, then there's $A \in |k\lambda D|$ w/ $\operatorname{mult}_x(A) > kn$; setting $\Delta = A/k$, we get $\Delta \sim_{\mathbb{Q}} \lambda D$, and $\operatorname{mult}_x(\Delta) > n$.

"pf": look at local coords:

$$\underbrace{\alpha}_{\substack{\text{one condition}\\ \text{to pass}\\ \text{thru } x}} + \underbrace{\beta_1 x_1 \dots + \beta_n x_n}_{\substack{n \text{ conditions}\\ \text{to have}\\ \text{mult.} \geq 2}} + \gamma_{11} x_1^2 + \gamma_{12} x_1 x_2 + \dots + \gamma_{nn} x_n^2 + \dots$$

Lemma: Let $x \in X$ be a smooth point, $\Delta \geq 0$ a $\mathbb{Q}$-divisor. If $\mathrm{mult}_x \Delta \geq \dim X$, then x is contained in an lc center of (X, Δ).

pf: blow up $x \in X$ & look at discrepancy.

Prop: Let X be a smooth variety, D a big divisor, and fix $0 < \lambda < 1$. Say that for $x \in X$ there's $D_x \sim_{\mathbb{Q}} \lambda D$ such that (X, Δ_x) has an isolated lc center at x. Then:

$$H^0(K_X + D) > 0 \quad \text{and} \quad h^0(K_X + 2D) \geq 2.$$

this will allow us to upgrade to a birational map.

Key point of proof: Nadel vanishing yields $\Rightarrow$ surj.

$$H^0(\mathcal{O}_X(K_X+D)) \longrightarrow H^0(\mathcal{O}_X(K_X+D)\otimes \mathcal{O}_X/\mathcal{J}), \quad] \ \text{copy of } \oplus\ \mathbb{C}.$$

w/ $\mathcal{J}$ some suitably chosen multiplier ideal.

Since x was an isolated lc center of (X, Δ_x),

it will be an isolated comp. of $\mathcal{O}_X/\mathcal{J}$.

Lemma: Say $\dim X = n$, and D is a divisor such that the induced rational map ϕ_D is generically finite. Then for $x \in X$ general, $\exists\ \Delta \sim nD$ such that

x is an isolated lc center of (X, Δ)

pf sketch: choose $x \in X$ s.t.

1) x is smooth

2) φ_D is defined & étale at x.

3) $x \notin \operatorname{Supp} \Delta$.

Let $z \in \mathbb{P}(H^0(D))$ the image of x, and choose $\tilde{H}_1, \dots, \tilde{H}_{n+1}$ n general hypersurfaces through z.

Let H_i = strict transform of $\tilde{H}_i$, and set $\Delta = \frac{n}{n+1}(H_1 + \dots + H_{n+1})$.

clearly x is an lc center of Δ, and x is isolated lc center.

Theorem: If $|sK_W|$ is birational for all W of general type of dim $< n$, then we can find a bound a s.t. $|rK_X|$ is birational, depending only on $\mathrm{vol}(X)$.

pf outline: Take $x \in X$ general, and choose $\lambda \in \mathbb{Q}$ s.t. $\lambda > \frac{n}{\mathrm{vol}(K_X)^{1/n}}$. We can produce a $\mathbb{Q}$-divisor $\Delta \sim_{\mathbb{Q}} \lambda K_X$ s.t. $\mathrm{mult}_x \Delta \geq n$. Then there is a lc center V of (X, Δ) containing x. Perturbing Δ slightly, we can assume V is an exc. lc center and there's a unique lc place E lying over V.

If $V = \{x\}$, we'd be done: a general pt $x \in X$ is an lc center of (X, Δ_x) for $\Delta_x \sim \lambda K_X$.

then we get a bound r' s.t. $\dim|rK_X| \geq 1$ for $r \geq r'$; the general fiber of a pencil $X \dashrightarrow \mathbb{P}^1$ is of general type

But there's no reason why $V = \{x\}$. So we proceed as follows: let $W \to V$ be a resolution of singularities.

By generality of $x \in X$, W is of general type.

By induction on dimension, $\exists\, \nu'$, independent of w, s.t. $|\nu' K_W|$ is birational. For general $w \in W$, we can produce a $\mathbb{Q}$-divisor $\Theta \sim \mu K_W$ $(\mu = (\dim W)\nu')$ such that w is an isolated lc center of (W, Θ).

<u>Goal:</u> want to "lift" w to $x' \in X$. That is, we want $\Delta' \sim \nu K_X$ (for ν controlled), such that x' is an isolated lc center of (X, Δ').

"image of $w \in W$.

<u>Theorem:</u> Retain our notation: X smooth dim n, $\Delta \sim \lambda K_X$, V an exc. lc. center of (X, Δ), $W \to V$ a resolution, W of general type. $\Theta \sim \mu K_W$ a divisor on W. Set $\nu = \lambda\mu + \lambda + \mu$.

There is a very general subset $U \subset V$, such that: if $W' \subset W$ is a log canonical center of (W, Θ) whose image V' intersects U, and $\delta > 0$, there's a divisor $\Delta' \sim (\nu + \delta) K_X$ such that V' is an exceptional lc center of (X, Δ').

Idea of proof: Let $Y \xrightarrow{\pi} (X, \Delta)$ be a log resolution, w/ E the unique prime divisor of discrepancy $= -1$ lying over V. I can assume that $E \xrightarrow{p} V$ dominates $W \to V$.

$$\begin{tikzcd}
mg^*\Theta & E_\xi \subset E \arrow[dl, "g"'] \arrow[r, hook] \arrow[dd, "p"] & Y \arrow[dd, "\pi"] \\
\Theta \subset W \arrow[u, "g"] \arrow[dr, "f"'] & & \\
& V \arrow[r, hook] & X
\end{tikzcd}$$

$\xi \in V$

Write: $K_Y + \Gamma = \pi^*(K_X + \Delta) + F$,

w/ • $\Gamma, F \geq 0$

• Γ, F share no common component.

Write $\Gamma = E + \Gamma^h + \Gamma^v$, $F = F^h + F^v$,

where "horizontal" components are those whose images contain V

and "vertical" components are everything else.

Since V was exc. lc center $\{\Gamma^h\} = \Gamma^h$ (i.e. $\lfloor \Gamma^h \rfloor = 0$)

$\Gamma^h|_{E_\xi} = (\Gamma - E)|_{E_\xi}$

$$(K_E + \Gamma^h_E)|_{E_\eta} = (K_Y + \Gamma)|_{E_\eta} = F|_{E_\eta} \geq 0$$

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + F$$

By log additivity of Kodaira dimension:

$h^0(E, m(K_{E/W} + \Gamma^h|_E) + H|_E) > 0$ for H an ample divisor on Y.

This gives an injection:

$$g^*|mK_W| \longrightarrow |m(K_E + \Gamma^h_E) + H|_E|$$

(induced by a section in $h^0(E, m(K_{E/W} + \Gamma^h|_E) + H|_E)$

$\Theta \sim mK_X$. Then: $g^*|m\ell K_W| \longrightarrow |\ell m(K_E + \Gamma^h_E) + H|_E|$.

Key lifting theorem: Let $E \subset Y$ be a smooth divisor in a smooth proj variety Y. Let Γ an snc divisor such that (Y, Γ) is lc and $\mathrm{coeff}_E(\Gamma) = 1$.

Let $C \geq 0$ be a $\mathbb{Q}$-divisor on Y not containing E.

H is an ample divisor, $A = (\dim Y) H$. Assume:

1) $K_E + (\Gamma - E)|_E$ is pseudoeff.

2) $\mathcal{O}_Y(m(K_Y + \Gamma + C))$ is generated by global sections at the generic pt of each lc center of $(Y, \lceil\Gamma\rceil)$

Then: the image of the restriction

$$H^0(Y, m(K_Y + \Gamma + C) + H + A) \longrightarrow H^0(E, m(K_E + (\Gamma - E)|_E + C|_E) + H|_E + A|_E)$$

contains the image of the inclusion

$$H^0(E, m(K_E + (\Gamma - E)|_E) + H|_E) \hookrightarrow$$

where $mg^* \Theta$ lives